Correlation functions for a Bose-Einstein condensate in the Bogoliubov approximation.

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(February 1, 2008)

In this article we introduce a differential equation for the first order correlation function $G^{(1)}$ of a Bose-Einstein condensate at $T = 0$. The Bogoliubov approximation is used. Our approach points out directly the dependence on the physical parameters. Furthermore it suggests a numerical method to calculate $G^{(1)}$ without solving an eigenvector problem. The $G^{(1)}$ equation is generalized to the case of non zero temperature.

I. INTRODUCTION

The observations of atomic Bose-Einstein condensate (BEC) in dilute atomic gases have triggered a great theoretical interest for this particular state of matter [1]. The Bose-Einstein condensate is a good opportunity to apply theoretically and verify experimentally the concepts of the quantum mechanics. In fact several interesting theoretical features, as macroscopic quantum tunneling and macroscopic quantum coherence, could be observed in BEC's in the near future. A condensate is characterized by a macroscopic occupation of a single particle state and by a large spatial correlation for the atomic spatial distribution. The long range spatial order has been studied in a series of theoretical papers [2–8]. On the experimental side interference experiments involving sodium and rubidium condensates [9,10] have demonstrated the presence of long-range order. The excellent agreement between the experimental results and theoretical analyses [6] has confirmed the presence of that long range order. More recent experiments have explored some features of second order [11] and third-order [12] atomic coherences . In ref. [11] the relationship between the second order coherence and the interaction energy has been studied, infering that release energy measurements are consistent with an unitary value for the second order coherence of a pure condensate. Burt et al. [12] have measured the three-body rubidium recombination rate of a condensate and of a cold noncondensate. They derive that the ratio of the third order coherences in those systems is 7.4 ± 2.6, in good agreement with the predicted value of 6.

Two standard tools to study the condensate are the Gross-Pitaevskii equation and the Bogoliubov approximation. Because in this approximation the hamiltonian is quadratic in the field, each property of the system is derivable by the mean field $\psi(\vec{x})$ and the first order correlation function $G^{(1)}(\vec{x},\vec{y})$, that is related to the first order coherence properties of a condensate. The mean field is described by the Gross-Pitaevskii equation. An extensive theoretical study of coherence properties of BEC has been performed by M. Naraschewski and R. J. Glauber [8]. To calculate the correlation functions they use the local density approximation, that is suitable for large enough systems. Furthermore they assume that the condensate kinetic energy is much smaller than the interaction energy. This condition is not fullfilled in a region close to the surface of the condensate, where the laplacian of the wave function, and therefore the kinetic energy, is not small. A standard way to calculate the correlation functions is to solve an eigenvector problem. For instance this method was used in [13] for a spherically symmetric harmonic-oscillator trap to evaluate the number of noncondensate atoms. In the anisotropic tridimensional case it is essential to choose a suitable set of functions to reduce the dimension of the matrix to be diagonalized. Often it is not easy to find this set and the matrix becomes very large for the numerical calculations, for example in the case of a double well trap.

Purpose of the present work is to find also for $G^{(1)}(\vec{x},\vec{y})$ a differential equation, similar to the Gross-Pitaevskii equation for $\psi(\vec{x})$, in order to provide the dependence of the first order correlation on the physical parameters. This equation suggests an alternative method to evaluate the correlation function. We introduce a differential equation for the 2×2 matrix $F(\vec{x},\vec{y}) = \langle \vec{x}|F|\vec{y}\rangle$, with $\vec{x}$ and $\vec{y}$ positions in the phase space. We find that the knowledge of $F(\vec{x},\vec{y})$ allows us to evaluate the correlation functions. The complete calculation of $G^{(1)}(\vec{x},\vec{y})$ is not much more efficient than the eigenvector evaluation. However our equation for $F(\vec{x},\vec{y})$ allows to obtain easily $G^{(1)}(\vec{x},\vec{y})$ for a fixed $\vec{y}$ or integrating it with a weight function $P(\vec{y})$. Our method is very useful if a complete information is not required. Moreover it is suitable to test numerically the approximations introduced with other methods of solutions. At first we will consider the case of zero temperature, then we generalize our equation to the case of non zero temperature [14].

II. CORRELATION FUNCTION

In the Bogoliubov approximation [15] the quantum boson field $\hat{\psi}(\vec{x})$ is written in the following way [16]

$$\hat{\psi}(\vec{x}) = \psi(\vec{x}) + \sum_{\lambda=1}^{\infty}[u_\lambda(\vec{x})\hat{a}_\lambda + v_\lambda^*(\vec{x})\hat{a}_\lambda^\dagger] \tag{1}$$

where ψ is determined by the time-independent Gross-Pitaevskii equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + g|\psi|^2\psi = \mu\psi. \tag{2}$$

and μ is the chemical potential. In Eq. (1) $\hat{a}_\lambda$ and $\hat{a}_\lambda^\dagger$ are annihilation and creation operators and (u_λ, v_λ) are the solutions of the following eigenvector problem

$$\mathcal{L}_\epsilon u_\lambda + g\psi^2 v_\lambda = E_\lambda u_\lambda \tag{3}$$
$$\mathcal{L}_\epsilon v_\lambda + g(\psi^*)^2 u_\lambda = -E_\lambda v_\lambda \tag{4}$$

with $\mathcal{L}_\epsilon = -\frac{\hbar^2}{2m}\nabla^2 + 2g|\psi|^2 + V - \mu + \epsilon$. ϵ is a positive infinitesimal number that we introduce to eliminate some divergences to be met with. To simplify the notation we do not indicate the dependence of the eigenvectors and the eigenvalues on ϵ. The zero energy eigenvector ($\lambda = 0$) is excluded in summation of Eq. (1)(as applied for instance in ref. [13]).

(u_λ, v_λ) satisfy the orthonormality and completeness relations

$$\int [u_\lambda(\vec{x})u_{\lambda'}^*(\vec{x}) - v_\lambda(\vec{x})v_{\lambda'}^*(\vec{x})]d^3x = \delta_{\lambda,\lambda'} \ \ , \ \ \forall\lambda,\lambda' \geq 0 \tag{5}$$

$$\sum_{\lambda=0}^{\infty}[u_\lambda(\vec{x})u_\lambda^*(\vec{y}) - v_\lambda(\vec{x})v_\lambda^*(\vec{y})] = \delta(\vec{x}-\vec{y}) \tag{6}$$

We point out that for $\epsilon = 0$ ($u_0 = \psi, v_0 = -\psi^*$) is the energy eigenvector with eigenvalue $E_0 = 0$. (u_0, v_0) is not normalizable, because $\int[|u_0|^2 - |v_0|^2]d^3x = \int[|\psi|^2 - |\psi|^2]d^3x = 0$.

The ground state is defined by the Eqs. $\hat{a}_\lambda|0>= 0$, $\forall\lambda \geq 1$. Explicitely using Eq. (1) we find that the first order correlation function for temperature $T = 0$ is given by

$$< \psi^\dagger(\vec{x})\psi(\vec{y}) >= \psi^*(\vec{x})\psi(\vec{y}) + \lim_{\epsilon\to 0^+}\sum_{\lambda,\lambda'=1}^{\infty} < 0|[u_\lambda^*(\vec{x})\hat{a}_\lambda^\dagger + v_\lambda(\vec{x})\hat{a}_\lambda][u_\lambda(\vec{y})\hat{a}_\lambda + v_\lambda^*(\vec{y})\hat{a}_\lambda^\dagger]|0 >$$

$$= \psi^*(\vec{x})\psi(\vec{y}) + C(\vec{x},\vec{y}) \tag{7}$$

With

$$C(\vec{x},\vec{y}) = \lim_{\epsilon\to 0^+}\sum_{\lambda=1}^{\infty} v_\lambda(\vec{x})v_\lambda^*(\vec{y}). \tag{8}$$

Eq. (1) cannot be considered a operator identity and, to be more rigorous, we should have followed the Gardiner's approach [17]. However the resulting Eqs. (7,8) are not changed. The set of Eqs. (3,4,7,8) defines completely our problem.

Our first purpose is to find for $C(\vec{x},\vec{y})$ a compact equation, where no eigenvector set appears and the dependence on the physical parameters is more evident. We introduce the annihilation operator field

$$\hat{\phi}(\vec{x}) = \sum_{\lambda=0}^{\infty}\left[u_\lambda(\vec{x})\hat{a}_\lambda + v_\lambda^*(\vec{x})\hat{a}_\lambda^\dagger\right], \tag{9}$$

where the summation is performed over all the eigenvectors. $\hat{\phi}(\vec{x})$ satisfies the usual commutation relations

$$[\hat{\phi}(\vec{x}), \hat{\phi}^\dagger(\vec{y})] = \delta(\vec{x}-\vec{y}) \tag{10}$$

We then consider the state $|\tilde{0}>$ defined by the equations $a_\lambda|\tilde{0}>= 0$, $\forall\lambda \geq 0$. It is evident that the Wigner function for $|\tilde{0}>$ is

$$W(\{a_\eta\},\{a_\eta^*\}) \propto e^{-2\sum_{\lambda=0}^{\infty} a_\lambda^* a_\lambda} \tag{11}$$

By the orthonormality relations (5) the Wigner function becomes

$$W(\{a_\eta\},\{a_\eta^*\}) \propto e^{-2\int d^3x \sum_{\lambda,\lambda'=0}^{\infty}(u_{\lambda'}u_\lambda^* - v_{\lambda'}v_\lambda^*)a_\lambda^* a_{\lambda'}}$$

$$= e^{-\int d^3x \sum_{\lambda,\lambda'=0}^{\infty}\left[(u_\lambda^* a_\lambda^* + v_\lambda a_\lambda)(u_{\lambda'}a_{\lambda'} - v_{\lambda'}^* a_{\lambda'}^*) + (u_\lambda^* a_\lambda^* - v_\lambda a_\lambda)(u_{\lambda'}a_{\lambda'} + v_{\lambda'}^* a_{\lambda'}^*)\right]} \tag{12}$$

It is useful to write a_λ as a two component vector and u_λ, v_λ as 2×2 matrices. We will use the notations

$$\vec{a}_\lambda = \begin{pmatrix} Re[a_\lambda] \\ Im[a_\lambda] \end{pmatrix}$$

$$\mathbf{u}_\lambda = \begin{pmatrix} Re[u_\lambda] & -Im[u_\lambda] \\ Im[u_\lambda] & Re[u_\lambda] \end{pmatrix},\ \mathbf{v}_\lambda = \begin{pmatrix} Re[v_\lambda] & -Im[v_\lambda] \\ Im[v_\lambda] & Re[v_\lambda] \end{pmatrix}$$

$$\vec{a}_\lambda^* = \hat{\sigma}_3 \vec{a}_\lambda \ , \quad \mathbf{u}^* = \hat{\sigma}_3 \mathbf{u}_\lambda \ , \quad \mathbf{v}_\lambda^* = \hat{\sigma}_3 \mathbf{v}_\lambda \tag{13}$$

where $\hat{\sigma}_3$ is the Pauli matrix with the diagonal elements 1 and -1. With this vector and matrix notations Eq. (12) becomes

$$W(\{\vec{a}_\eta\}) \propto e^{-2\int d^3x \sum_{\lambda,\lambda'=0}^{\infty}(\vec{a}_\lambda^\dagger \mathbf{u}_\lambda^\dagger + \vec{a}_\lambda^{*\dagger}\mathbf{v}_\lambda^{*\dagger})(\mathbf{u}_{\lambda'}\vec{a}_{\lambda'} - \mathbf{v}_{\lambda'}^*\vec{a}_{\lambda'}^*)} \tag{14}$$

Eqs. (3,4) allow us to find that

$$H_1(\mathbf{u}\vec{a} + \mathbf{v}^*\vec{a}^*) = E_\lambda(\mathbf{u}\vec{a} - \mathbf{v}^*\vec{a}^*) \tag{15}$$

$$H_2(\mathbf{u}\vec{a} - \mathbf{v}^*\vec{a}^*) = E_\lambda(\mathbf{u}\vec{a} + \mathbf{v}^*\vec{a}^*) \tag{16}$$

where $H_1 = \mathcal{L}_\epsilon + g\Psi^2\hat{\sigma}_3$, $H_2 = \mathcal{L}_\epsilon - g\Psi^2\hat{\sigma}_3$ and Ψ is a 2×2 matrix constructed by ψ as the $\mathbf{u}$ and $\mathbf{v}$ matrices. ¿From these equations we deduce

$$(H_1 \cdot H_2)^{1/2}(\mathbf{u}\vec{a} - \mathbf{v}^*\vec{a}^*) = E_\lambda(\mathbf{u}\vec{a} - \mathbf{v}^*\vec{a}^*) \tag{17}$$

and from Eqs. (15,17)

$$W(\{\vec{a}_\eta\}) \propto e^{-2\int d^3x \sum_{\lambda,\lambda'=0}^{\infty}(\vec{a}_\lambda^\dagger \mathbf{u}_\lambda^\dagger + \vec{a}_\lambda^{*\dagger}\mathbf{v}_\lambda^{*\dagger})(H_1\cdot H_2)^{-1/2}H_1(\mathbf{u}_{\lambda'}\vec{a}_{\lambda'} + \mathbf{v}_{\lambda'}^*\vec{a}_{\lambda'}^*)}. \tag{18}$$

Note that if we did not use our real notation we had to introduce antilinear operators.

We now perform the transformation

$$\vec{\phi}(\vec{x}) = \sum_{\lambda=0}^{\infty}[\mathbf{u}_\lambda(\vec{x})\vec{a}_\lambda + \mathbf{v}_\lambda^*(\vec{x})\vec{a}_\lambda^*] \tag{19}$$

to obtain the W as a function of the field $\vec{\phi}(\vec{x})$ that corresponds to the quantum field $\hat{\phi}(\vec{x})$ of Eq. (9)

$$W(\{\vec{\phi}\}) \propto e^{-2\int d^3x \vec{\phi}^\dagger(\vec{x})(H_1\cdot H_2)^{-1/2}H_1\vec{\phi}(\vec{x})} \tag{20}$$

It is evident that $(H_1\cdot H_2)H_1 = H_1(H_2\cdot H_1)$. Therefore $M = (H_1\cdot H_2)^{-1/2}H_1 = H_1(H_2\cdot H_1)^{-1/2} = M^\dagger$, *i.e.* M is a symmetric operator. More in general

$$f(H_1\cdot H_2)\cdot H_1 = H_1 \cdot f(H_2\cdot H_1) \tag{21}$$

It is then easy to demonstrate that the mean weighted with the Wigner function is given by

$$< \phi_i(\vec{x})\phi_j(\vec{y}) >_W = \frac{1}{4}M^{-1}_{(\vec{x},i),(\vec{y},j)} \equiv \frac{1}{4}\left[H_1^{-1}(H_1\cdot H_2)^{1/2}\right]_{(\vec{x},i),(\vec{y},j)} = \frac{1}{4}\langle \vec{x},i|H_1^{-1}(H_1\cdot H_2)^{1/2}|\vec{y},j\rangle. \tag{22}$$

In fact, if $\hat{T}_{k,l}$ is a symmetric matrix then exists a orthogonal transformation $z_k = \sum_l \hat{O}_{k,l} Z_l$ that diagonalizes $\hat{T}$. Therefore

$$\int z_i z_j e^{-2\sum_{k,l} \hat{T}_{k,l} z_k z_l} d\vec{z} = \sum_{i',j'} \hat{O}_{i,i'} \hat{O}_{j,j'} \int Z_{i'} Z_{j'} e^{-2\sum_k \hat{T}'_{k,k} Z_k Z_k} d\vec{Z}$$

$$= 1/4 \sum_{i',j'} O_{i,i'} O_{j,j'} (\hat{T}'^{-1})_{i',j'} = 1/4 (\hat{T}^{-1})_{i,j} \tag{23}$$

The expectation value of an operator $F(\hat{\phi}, \hat{\phi}^\dagger)$, symmetrically ordered, is given by the mean of the classical function $F(\phi, \phi^*)$ weighed with the Wigner function, therefore

$$1/2 < 0|\hat{\phi}^\dagger(\vec{x})\hat{\phi}(\vec{y}) + h.c|0 >=< (\phi_1(\vec{x}) - i\phi_2(\vec{x}))(\phi_1(\vec{y}) + i\phi_2(\vec{y})) >_W \tag{24}$$

Combining Eq. (10,22,24) we find that

$$\tilde{C}(\vec{x}, \vec{y}) \equiv < 0|\hat{\phi}^\dagger(\vec{x})\hat{\phi}(\vec{y})|0 >= F_{(\vec{x},1),(\vec{y},1)} + F_{(\vec{x},2),(\vec{y},2)} + iF_{(\vec{x},1),(\vec{y},2)} - iF_{(\vec{x},2),(\vec{y},1)} \tag{25}$$

where the operator F is defined by

$$F \equiv \frac{1}{4} H_1^{-1} \left[(H_1 \cdot H_2)^{1/2} - H_1 \right]$$

The $\lambda = 0$ term should be subtracted from $\tilde{C}(\vec{x}, \vec{y})$ in order to obtain the $C(\vec{x}, \vec{y})$ quantity defined in Eq. (8)

$$C(\vec{x}, \vec{y}) = \lim_{\epsilon \to 0^+} \left[\tilde{C}(\vec{x}, \vec{y}) - v_0(\vec{x}) v_0^*(\vec{y}) \right] \tag{26}$$

where $\tilde{C}(\vec{x}, \vec{y})$ and $v_0(\vec{x})$ depend implicitly by the parameter ϵ. $v_0(\vec{x})$ can be calculated solving the dynamical equations obtained replacing E_λ with $i\hbar\frac{\partial}{\partial t}$ in Eqs. (3,4). In fact, if $u(\vec{x}, t)$ and $v(\vec{x}, t)$ are the solution of these equations then

$$v_0(\vec{x}) \propto \int_{-\infty}^{\infty} dt \int_0^\eta dE e^{iEt} v(\vec{x}, t) = \int_{-\infty}^{\infty} dt \frac{1}{it} \left(e^{i\eta t} - 1 \right) v(\vec{x}, t)$$

$$u_0(\vec{x}) \propto \int_{-\infty}^{\infty} dt \int_0^\eta dE e^{iEt} u(\vec{x}, t) = \int_{-\infty}^{\infty} dt \frac{1}{it} \left(e^{i\eta t} - 1 \right) u(\vec{x}, t) \tag{27}$$

where η is a number such that $E_0 < \eta < E_1$. It is convenient to get $u(\vec{x}, 0) = v(\vec{y}, 0) = \psi$ as initial state and to introduce a temporal gaussian window to lower the convergence time.

We have reduced our problem to the evaluation of the operator F. F satisfies the following equation

$$H_1 \cdot F = \frac{1}{4} \left[(H_1 \cdot H_2)^{1/2} - H_1 \right] \equiv S \tag{28}$$

that is

$$-\nabla^2 F(\vec{x}, \vec{y}) + M^2 F(\vec{x}, \vec{y}) = \frac{2m}{\hbar^2} S(\vec{x}, \vec{y}) \tag{29}$$

where

$$M^2 = \frac{2m}{\hbar^2} \left[V + 2g\Psi^*\Psi - g\Psi^2 - \mu + \epsilon \right] \tag{30}$$

Eq. (29) is a Yukawa-like equation with a coordinate dependent mass and a charge distribution $S(\vec{x}, \vec{y})$ in $\vec{x}$. Both S and M are 2×2 matrices.

If S is known, $F(\vec{x}, \vec{y})$ can be evaluated finding for all the $\vec{y}$ positions the stationary state of the following differential equation:

$$\frac{\partial}{\partial t} F(\vec{x}, \vec{y}) + H_1 F(\vec{x}, \vec{y}) = S(\vec{x}, \vec{y}). \tag{31}$$

Every function whose evolution is determined by Eq (31) collapses in this stationary state because H_1 is a positive eigenvalue operator.

The standard method to calculate the correlation function is to solve the eigenvector problem of Eqs. (3,4) using Eqs. (7,8). However in some cases the matrix to be diagonalized becomes too large to be handled. Eq. (28) can be useful to extract informations about the correlation function without the resolution of an eigenvector problem.

If we want to calculate $C(\vec{x},\vec{y})$ with a fixed $\vec{y}$ or integrating $\vec{y}$ with a weight function $P(\vec{y})$

$$C(\vec{x}) = \int C(\vec{x},\vec{y})P(\vec{y})d^3y \tag{32}$$

we have to solve only the differential equation (31) with $\vec{y}$ fixed or with the source term

$$(\vec{S}_0)_i(\vec{x}) = \sum_{j=1}^{2} \int S_{i,j}(\vec{x},\vec{y})\vec{P}_j(\vec{y})d^3y. \tag{33}$$

where $\vec{P}(\vec{y}) = \begin{pmatrix} P(\vec{y}) \\ P(\vec{y}) \end{pmatrix}$. This approach allows to decrease considerably the computation time.

The question to be solved is the evaluation of $\vec{S}_0$. To calculate the source term the square root in the second term of Eq. (28) should be known, but that requires to solve an eigenvector problem that could be avoided through an alternative method. It is well known from the Dirac theory that a square root of the operator $-\nabla^2 + m^2$ is a local operator with first order differential derivatives that multiply anticommuting matrices. The only difference between that square root and the nonlocal operator $\sqrt{-\nabla^2 + m^2}$ is the sign of the eigenvalues, that in the last case are all positive.

A local non-positive square root exists also for $R = H_1 \cdot H_2$. R has the form, apart from a constant factor,

$$R = \left[-\nabla^2 + Q_1 + Q_2\hat{\sigma}_1 + Q_3\hat{\sigma}_3\right]\left[-\nabla^2 + Q_1 - Q_2\hat{\sigma}_1 - Q_3\hat{\sigma}_3\right] \tag{34}$$

where Q_1, Q_2 and Q_3 are three real functions. It is easy to demonstrate that the operator

$$H_r = \hat{\sigma}_2(-\nabla^2 + Q_1) - i\hat{\sigma}_3 Q_2 + i\hat{\sigma}_1 Q_3 \tag{35}$$

is square root of R, that is $H_r^2 = R$.

If $\vec{P}_+$ and $\vec{P}_-$ are the projections of $\vec{P}$ respectively over the positive and negative eigenvalue subspaces of H_r then

$$\vec{S}_0(\vec{x}) = H_r(\vec{P}_+ - \vec{P}_-) \tag{36}$$

We now describe how to handle $\vec{P}_+ - \vec{P}_-$. If $\vec{P}(\vec{x},\tau)$ is solution of the equation

$$i\frac{\partial}{\partial\tau}\vec{P} = H_r\vec{P} \tag{37}$$

it is evident that

$$\int_0^\infty dE \int_{-\infty}^{\infty} \vec{P}(\vec{x},\tau)e^{iE\tau}d\tau = \vec{P}_+(\vec{x})$$
$$\int_{-\infty}^0 dE \int_{-\infty}^{\infty} \vec{P}(\vec{x},\tau)e^{iE\tau}d\tau = \vec{P}_-(\vec{x}) \tag{38}$$

Performing the integration in E, we obtain

$$\vec{P}_+(\vec{x}) - \vec{P}_-(\vec{x}) = \frac{1}{\pi}\lim_{\eta\to 0^+}\int_\eta^\infty \frac{1}{\tau}[\vec{P}(\vec{x},\tau) - \vec{P}(\vec{x},-\tau)]d\tau \tag{39}$$

Therefore

$$\vec{S}_0(\vec{x}) = \frac{1}{\pi}H_r\lim_{\eta\to 0^+}\int_\eta^\infty \frac{1}{\tau}[\vec{P}(\vec{x},\tau) - \vec{P}(\vec{x},-\tau)]d\tau \tag{40}$$

The solution of Eq. (37) allows to evaluate the source term of Eq. (33). The direct calculation of Eq. (39) is probably not the best choice. In fact, if the ratio between the greatest and lowest frequencies is too large then the integration step has to be too small with respect to the integration time. In this case it is convenient to perform the energy integration of Eqs. (38) over the windows (E_1, E_2), (E_2, E_3), (E_3, E_4), ..., with $E_1 > E_2 > E_3$... and to choose for each window a suitable integration step. It is also convenient to use a temporal gaussian window to reduce the calculation time. In this article we do not discuss these numerical questions into details.

III. NUMERICAL TESTS

At this stage we have all the tools to evaluate the correlation function of Eq. (32). We have checked numerically in the one-dimensional case that the same source terms are obtained from Eq. (40) and by the diagonalization. Also the validity of Eq. (28) has been verified numerically.

In order to test the technique we have considered the case of a one-dimensional harmonic trap with $V(x) = 1/2x^2$ and a coupling constant $g = 10$ ($\hbar = m = 1$). The $\psi(x)$ solution of the Gross-Pitaevskii equation is reported in the lower part of Fig. 1. Instead sections of $S_{1,1}(x, y)$ are reported in the upper part of the figure for different values of y. The results of the figure point out that the source term is a near diagonal operator. In fact S is a sum of a diagonal matrix and a smooth function $f(x, y)$. In other terms, for every y, the source has a point-like charge with a cloud around.

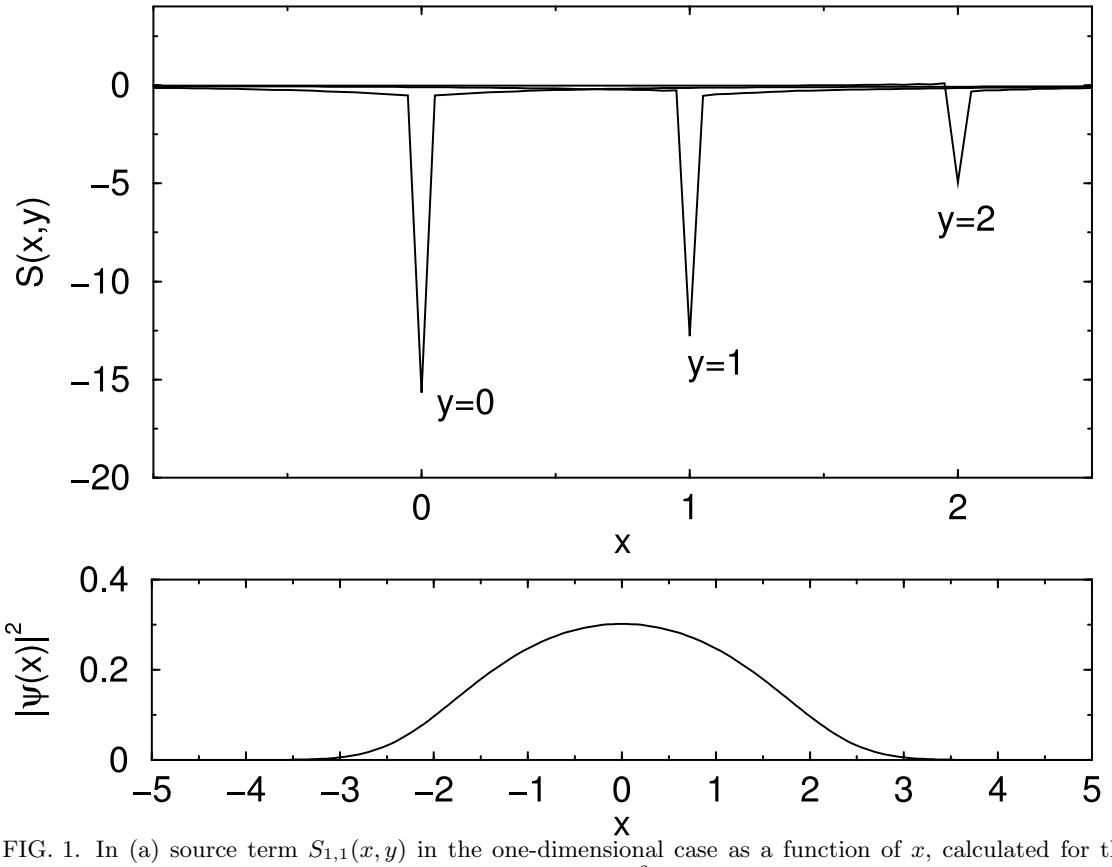

FIG. 1. In (a) source term $S_{1,1}(x, y)$ in the one-dimensional case as a function of x, calculated for three values of y and a coupling constant $g = 10$. In (b) density function $|\psi(x)|^2$ from the Gross-Pitaevskii equation for the same parameters. Adimensional unities are used.

We have considered also a tridimensional case. The studied system is constituted by ^{87}Rb atoms confined in a spherical harmonic trap in the $|F = 1, m_f = -1\rangle$ hyperfine sublevels. For the scattering length we have used $a = 109.1$ a.u.. The trap frequency is supposed $\omega = 2\pi \cdot 300\ \mathrm{s}^{-1}$. We have imposed $\vec{y} = 0$ in Eq.(32) in order to exploit the trap symmetry and therefore to simplify the calculation. The application into the case of asymmetric trap require only some additional algebra [18]. In Fig. 2 we plot $C(r) = C(\vec{x}, \vec{y} = 0)$ as a function of $r = |\vec{x}|$ for different values of the boson number N. The functions obtained by diagonalization and solving our differential equation overlap and therefore are indistinguishable in the plot.

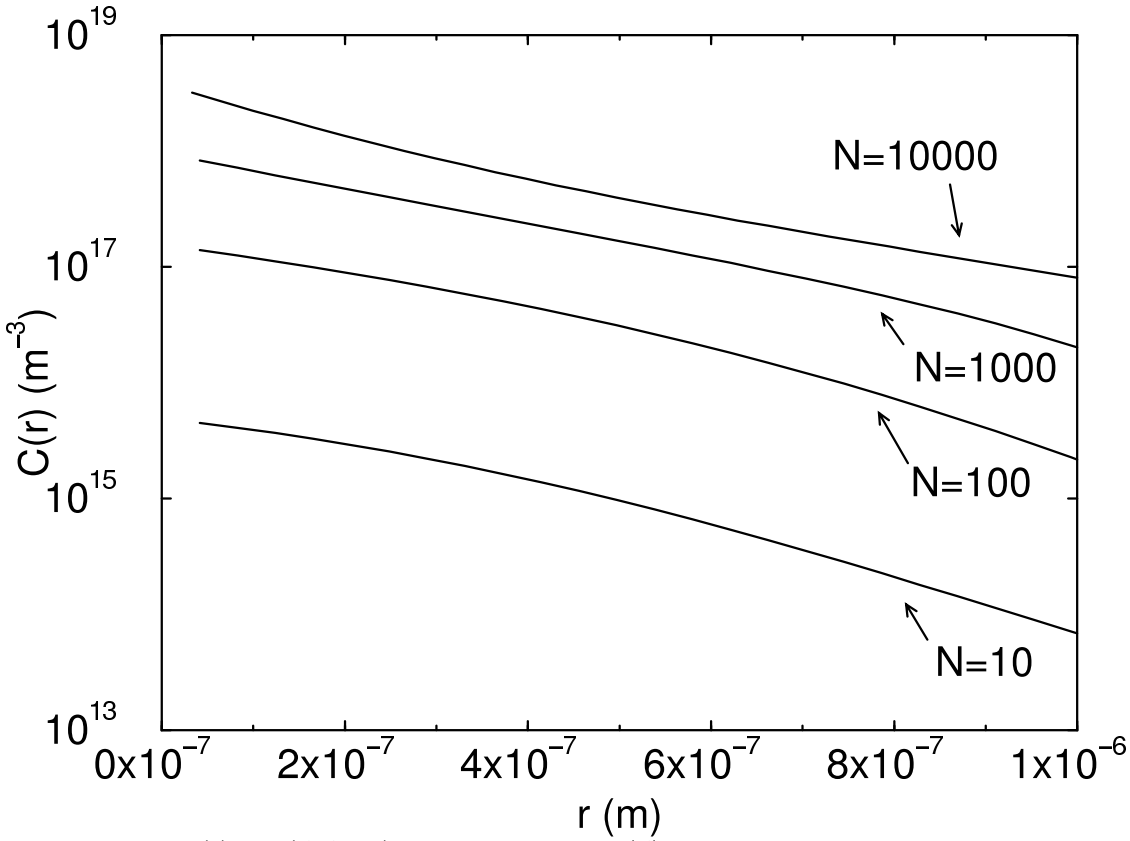

FIG. 2. Plot of $C(r) = C(\vec{x}, \vec{y} = 0)$ as a function of $r = |\vec{x}|$ for some values of the boson number N. We have considered ^{87}Rb atoms trapped in the $|F = 1, M_m = -1\rangle$ hyperfine sublevel.

We note that $C(r = 0)$ increases with N. This is obvious because the correlations of the field fluttuations are a consequence of the Gross-Pitaevskii non-linear term. The variation scale of $C(r)$ is given by $1/M(r)$ and for $N = 0$ its magnitude is of the order of $sqrt\hbar/m\omega = 6.2 \cdot 10^{-7} m$.

IV. FINITE TEMPERATURE

Eq. (28) can be generalized to include the finite temperature fluctuations. Eq. (11) is replaced by

$$W(\{a_\eta\}, \{a_\eta^*\}) \propto e^{-\sum_{\lambda=0}^{\infty} \frac{a_\lambda^* a_\lambda}{1/2 + \left(e^{\beta_\lambda} - 1\right)^{-1}}} \tag{41}$$

where $\beta_\lambda = E_\lambda / kT$. For $T \gg E_\lambda$ the correct classical distribution is obtained.

Using Eq. (17) we find W as a function of $\vec{\phi}$

$$W(\{\vec{\phi}\}) \propto e^{-2 \int d^3x \vec{\phi}^\dagger(\vec{x}) (H_1 \cdot H_2)^{-1/2} A_T^{-1} H_1 \vec{\phi}(\vec{x})} \tag{42}$$

where A_T is the operator

$$A_T = 1 + 2 \left(e^{\frac{(H_1 \cdot H_2)^{1/2}}{kT}} - 1 \right)^{-1} \tag{43}$$

The F operator of Eq. (25) is replaced by

$$F_T \equiv \frac{1}{4} H_1^{-1} \left[A_T \cdot (H_1 \cdot H_2)^{1/2} - H_1 \right] \tag{44}$$

It is possible to find a relation between F_T and $F = F_0$. We subtract $1/4$ from the two terms of Eq. (44) and multiply them by A_T^{-1}. We obtain using Eq. (21)

$$H_1 \cdot B_T^{-1}(F_T - 1) = \frac{1}{4} \cdot (H_1 \cdot H_2)^{1/2}. \tag{45}$$

where

$$B_T = 1 + 2\left(e^{\frac{(H_2 \cdot H_1)^{1/2}}{kT}} - 1\right)^{-1}. \tag{46}$$

Therefore

$$B_T^{-1}(F_T - 1) = B_{T'}^{-1}(F_{T'} - 1) \tag{47}$$

Setting $T' = 0$ we finally find

$$F_T = 1 + B_T \cdot (F - 1). \tag{48}$$

We can perform the following expansion

$$B_T = 1 + 2e^{-\frac{(H_2 \cdot H_1)^{1/2}}{kT}}\left(1 - e^{-\frac{(H_2 \cdot H_1)^{1/2}}{kT}}\right)^{-1}$$

$$= 1 + 2\sum_{n=1}^{\infty} e^{-\frac{n(H_2 \cdot H_1)^{1/2}}{kT}}. \tag{49}$$

If $\vec{F}_i(\vec{x}) = \sum_j \int F_{i,j}(\vec{x},\vec{y})\vec{P}_j(\vec{y})d^3y$ and $\vec{F}_T(\vec{x}) = \sum_j \int (F_T)_{i,j}(\vec{x},\vec{y})\vec{P}_j(\vec{y})d^3y$ then

$$\vec{F}_T = \vec{F} + 2\sum_{n=1}^{\infty} e^{-\frac{n(H_2 \cdot H_1)^{1/2}}{kT}}(\vec{F} + \vec{P})$$

$$= \vec{F} + 2\sum_{n=1}^{\infty} e^{-\frac{n\tilde{H}_r}{kT}}(\vec{F}_+ + \vec{P}_+) + 2\sum_{n=1}^{\infty} e^{\frac{n\tilde{H}_r}{kT}}(\vec{F}_- + \vec{P}_-) \tag{50}$$

where $\vec{F}_+$, $\vec{P}_+$ and $\vec{F}_-$, $\vec{P}_-$ are the projections over the positive and negative eigenvalues subspaces of $\tilde{H}_r = \hat{\sigma}_3 H_r \hat{\sigma}_3$. If $\vec{F}_\pm(\vec{x},\tau)$ and $\vec{P}_\pm(\vec{x},\tau)$ are the solution of the differential equation [19]

$$\frac{\partial}{\partial\tau}(\cdot) = \tilde{H}_r(\cdot) \tag{51}$$

with $\vec{F}_\pm(\vec{x},0) = \vec{F}_\pm$ and $\vec{P}_\pm(\vec{x},0) = \vec{P}_\pm$, then

$$\vec{F}_T(\vec{x}) = \vec{F}(\vec{x}) + 2\sum_{n=1}^{\infty}\left[(\vec{F}_+ + \vec{P}_+)(\vec{x},-\frac{n}{kT}) + (\vec{F}_- + \vec{P}_-)(\vec{x},\frac{n}{kT})\right] \tag{52}$$

Equations similar to the (25,26) ones can be defined for temperatures T different from zero. Therefore we have shown that it is possible derive the correlation function for $T \neq 0$ by knowing it for $T = 0$.

V. CONCLUSION

In conclusion we have introduced a differential equation that is useful to evaluate numerically the first order correlation function $G^{(1)}(\vec{x},\vec{y})$ for some fixed $\vec{y}$ or its integration over $\vec{y}$ with a weight function $P(\vec{y})$. In the Bogoliubov approximation the hamiltonian is quadratic in the field, therefore the ground state is a squeezed state, that is, the Wigner function is a gaussian one for all its infinite modes. The gaussian parameters are $\psi(\vec{x})$, that defines its position in phase space, and the function $F(\vec{x},\vec{y})$ that we have introduced. There are not other free parameters. Therefore each property of the BEC is derivable by solving the Gross-Pitaevskii equation and Eq. (28). In particular the resolution of these equations allows to evaluate the higher order correlation functions.

Our method can be applied to study the quantum fluttuations of the condensate in double-well traps, improving the two mode model, that is the standard approach to deal with these problems [20].

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